HIGHER LAME EQUATIONS AND CRITICAL POINTS OF MASTER FUNCTIONS

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Abstract. Under certain conditions, we give an estimate from above on the number of differential equations of order $r+1$ with prescribed regular singular points, prescribed exponents at singular points, and having a quasi-polynomial flag of solutions. The estimate is given in terms of a suitable weight subspace of the tensor power $U(\mathfrak{n}_-)^{\otimes(n-1)}$, where n is the number of singular points in $\mathbb{C}$ and $U(\mathfrak{n}_-)$ is the enveloping algebra of the nilpotent subalgebra of $\mathfrak{gl}_{r+1}$.

Dedicated to Askold Khovanskii on the occasion of his 60^{th} birthday.

1. Introduction

Consider the differential equation

$$F(x)\,u''(x) \;+\; G(x)\,u'(x) \;+\; H(x)\,u(x) \;=\; 0\,, \tag{1}$$

where $F(x)$ is a polynomial of degree n, and $G(x)$, $H(x)$ are polynomials of degree not greater than $n-1$, $n-2$, respectively. If $F(x)$ has no multiple roots, then all singular points of the equation are regular singular. Write

$$F(x) \;=\; \prod_{s=1}^{n}(x-z_s)\,, \qquad \frac{G(x)}{F(x)} \;=\; -\sum_{s=1}^{n}\frac{m_s}{x-z_s} \tag{2}$$

for suitable complex numbers m_s , z_s . Then 0 and m_s+1 are exponents at z_s of equation (1). If $-l$ is one of the exponents at ∞ , then the other is $l-1-\sum_{s=1}^{n} m_s$.

Problem ([Sz], Ch. 6.8) *Given polynomials $F(x)$, $G(x)$ as above and a non-negative integer l,*

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(a) *find a polynomial $H(x)$ of degree at most $n-2$ such that equation (1) has a polynomial solution of degree l;*
(b) *find the number of solutions to Problem (a).*

If $H(x), u(x)$ is a solution to Problem (a), then the corresponding equation (1) is called *a Lame equation* and the polynomial $u(x)$ is called *a Lame function.*

Example. Let $F(x)=1-x^2$, $G(x)=\alpha-\beta+(\alpha+\beta+2)x$. Then $H=l(l+\alpha+\beta+1)$ and the corresponding polynomial solution of degree l, normalized by the condition $u(1)=\binom{l+\alpha}{l}$, is called *the Jacobi polynomial* and denoted by $P_l^{(\alpha,\beta)}(x)$.

The following result is classical.

Theorem 1.1 (Cf. [Sz], Ch. 6.8, [St])**.**

- *Let $u(x)$ be a polynomial solution of (1) of degree l with roots $t_1^0,\dots,t_l^0$ of multiplicity one. Then $\boldsymbol{t}^0=(t_1^0,\dots,t_l^0)$ is a critical point of the function*

$$\Phi_{l,n}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{m}) \;=\; \prod_{i=1}^{l}\prod_{s=1}^{n}(t_i-z_s)^{-m_s}\prod_{1\le i<j\le l}(t_i-t_j)^2\ ,$$

  where $\boldsymbol{z}=(z_1,\dots,z_n)$ and $\boldsymbol{m}=(m_1,\dots,m_n)$.
- *Let $\boldsymbol{t}^0$ be a critical point of the function $\Phi_{l,n}(\,\cdot\,;\boldsymbol{z};\boldsymbol{m})$, then the polynomial $u(x)$ of degree l with roots $t_1^0,\dots,t_l^0$ is a solution of (1) with $H(x)=(-F(x)u''(x)-G(x)u'(x))/u(x)$ being a polynomial of degree at most $n-2$.*

The function $\Phi_{l,n}(\,\cdot\,;\boldsymbol{z};\boldsymbol{m})$ is called *the master function.* The master function is a symmetric function of the variables $t_1,\dots,t_l$. Therefore, the symmetric group S_l naturally acts on the set of critical points of the master function by permuting the coordinates.

By Theorem 1.1, the S_l-orbits of critical points are in one-to-one correspondence with solution $H(x),u(x)$ of Problem (a) such that $u(x)$ has no multiple roots.

The following result is also classical.

Theorem 1.2 (Cf. [Sz], Ch. 6.8, [H], [St])**.** *If $z_1,\dots,z_n$ are distinct real numbers and $m_1,\dots,m_n$ are negative numbers, then the number of solutions to Problem (a) is equal to $\binom{l+n-2}{l}$.*

Under these conditions on $\boldsymbol{z}$ and $\boldsymbol{m}$, the master function has exactly $\binom{l+n-2}{l}$ S_l-orbits of critical points, see [Sz] and [V3].

The number $\binom{l+n-2}{l}$ has the following representation theoretical interpretation. The universal enveloping algebra $U(\mathfrak{n}_-)$ of the nilpotent subalgebra $\mathfrak{n}_-\subset\mathfrak{gl}_2$ is generated by one element e_{21} and is weighted by powers of the generator. The number $\binom{l+n-2}{l}$ is the dimension of the weight l part of the tensor power $U(\mathfrak{n}_-)^{\otimes(n-1)}$.

The case of nonnegative integers $m_1, \dots, m_n$ is interesting for applications to the Bethe ansatz method in the Gaudin model. For a nonnegative integer m, let L_m be the $m+1$-dimensional irreducible $\mathfrak{sl}_2$-module. If $m_1, \dots, m_n$ are nonnegative integers and $m_\infty = \sum_{s=1}^n m_s - 2l$ is a nonnegative integer, then for any distinct $z_1, \dots, z_n$ the number of S_l-orbits of critical points of the master function function $\Phi_{l,n}(\,\cdot\,; \boldsymbol{z}; \boldsymbol{m})$ is not greater than the multiplicity of the $\mathfrak{sl}_2$-module L_{m_∞} in the tensor product $\otimes_{s=1}^n L_{m_s}$. Moreover, for generic $z_1, \dots, z_n$, the number of S_l-orbits is equal to that multiplicity, see [ScV].

The goal of this paper is to generalize the formulated results. We consider linear differential equations of order $r+1$ with regular singular points only, located at $z_1, \dots, z_n, \infty$, and having prescribed exponents at each of the singular points. We introduce the notion of a quasi-polynomial flag of solutions for such a differential equation. (For a second order differential equation, a quasi-polynomial flag is just a solution of the form $y(x) \prod_{s=1}^n (x - z_s)^{\lambda_s}$ where $y(x)$ is a polynomial and $\lambda_1, \dots, \lambda_n$ are some complex numbers.) We prove two facts:

- Differential equations with a quasi-polynomial flag, prescribed singular points and exponents are in one-to-one correspondence with suitable orbits of critical points of some $\mathfrak{gl}_{r+1}$-master function.
- Under explicit generic conditions on exponents (they must be *separated*), we show that the number of orbits of critical points of the corresponding master function is not greater than the dimension of a suitable weight subspace of $U(\mathfrak{n}_-)^{\otimes(n-1)}$, where $\mathfrak{n}_-$ is the nilpotent subalgebra of $\mathfrak{gl}_{r+1}$.

The proofs are based on results from [MV2, BMV].

This paper was motivated by discussions with B. Shapiro of his preprint [BBS] in which another generalization of Problems (a) and (b) is introduced for differential equations of order $r+1$. The authors thank B. Shapiro for useful discussions of his preprint.

2. Critical points of $\mathfrak{gl}_{r+1}$-master functions

2.1. **Lie algebra $\mathfrak{gl}_{r+1}$.** Consider the Lie algebra $\mathfrak{gl}_{r+1}$ with standard generators $e_{a,b}$, $a, b = 1, \dots, r+1$, and Cartan decomposition $\mathfrak{gl}_{r+1} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$,

$$\mathfrak{n}_- = \oplus_{a>b} \mathbb{C} \cdot e_{a,b}\ , \qquad \mathfrak{h} = \oplus_{a=1}^{r+1} \mathbb{C} \cdot e_{a,a}\ , \qquad \mathfrak{n}_+ = \oplus_{a<b} \mathbb{C} \cdot e_{a,b}\ .$$

Set $h_a = e_{a,a} - e_{a+1,a+1}$ for $a = 1, \dots, r$. Let $e^*_{1,1}, \dots, e^*_{r+1,r+1} \in \mathfrak{h}^*$ be the basis dual to the basis $e_{1,1}, \dots, e_{r+1,r+1} \in \mathfrak{h}$. Set $\alpha_a = e^*_{a,a} - e^*_{a+1,a+1}$ for $a = 1, \dots, r$. Fix the scalar product on $\mathfrak{h}^*$ such that $(e^*_{a,a}, e^*_{b,b}) = \delta_{a,b}$.

Let $U(\mathfrak{n}_-)$ be the universal enveloping algebra of $\mathfrak{n}_-$,

$$U(\mathfrak{n}_-) \ = \ \oplus_{\boldsymbol{l} \in \mathbb{Z}^r_{\geq 0}} U(\mathfrak{n}_-)[\boldsymbol{l}]\ ,$$

where for $\boldsymbol{l} = (l_1, \dots, l_r)$, the space $U(\mathfrak{n}_-)[\boldsymbol{l}]$ consists of elements f such that

$$[f, h] \;=\; \langle h\,,\, \sum_{i=1}^{r} l_i \alpha_i \rangle f \;.$$

The element $\prod_i e_{a_i,b_i}$ with $a_i > b_i$ belongs to the graded subspace $U(\mathfrak{n}_-)\,[\boldsymbol{l}]$, where $\boldsymbol{l} = \sum_i \boldsymbol{l}_i$ with

$$\boldsymbol{l}_i \;=\; (0, 0, \dots, 0, 1_b, 1_{b+1}, \dots, 1_{a-1}, 0, 0, \dots, 0)\;.$$

Choose an order on the set of elements $e_{a,b}$ with $r+1 \geq a > b \geq 1$. Then the ordered products $\prod_{a>b} e_{a,b}^{n_{a,b}}$ form a graded basis of $U(\mathfrak{n}_-)$.

The grading of $U(\mathfrak{n}_-)$ induces the grading of $U(\mathfrak{n}_-)^{\otimes k}$ for any positive integer k,

$$U(\mathfrak{n}_-)^{\otimes k} \;=\; \oplus_{\boldsymbol{l}}\, U(\mathfrak{n}_-)^{\otimes k}[\boldsymbol{l}]\;.$$

Denote

$$d(k, \boldsymbol{l}) \;=\; \dim\; U(\mathfrak{n}_-)^{\otimes k}[\boldsymbol{l}]\;.$$

For a weight $\Lambda \in \mathfrak{h}^*$, denote by L_Λ the irreducible $\mathfrak{gl}_{r+1}$-module with highest weight Λ. Let

$$\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)\;, \qquad \Lambda_s \in \mathfrak{h}^*\;,$$

be a collection of weights and $\boldsymbol{l} = (l_1, \dots, l_r)$ a collection of nonnegative integers. Let

$$L_{\boldsymbol{\Lambda}} \;=\; L_{\Lambda_1} \otimes \dots \otimes L_{\Lambda_n}$$

be the tensor product of irreducible $\mathfrak{gl}_{r+1}$-modules. Let $L_{\boldsymbol{\Lambda}} \;=\; \oplus_{\boldsymbol{l} \in \mathbb{Z}^r_{\geq 0}} L_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$ be its weight decomposition, where $L_{\boldsymbol{\Lambda}}[(l_1, \dots, l_r)]$ is the subspace of vectors of weight $\sum_{s=1}^n \Lambda_s - \sum_{i=1}^r l_i \alpha_i$. Let $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{l}] \subset L_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$ be the subspace of singular vectors, i.e. the subspace of vectors annihilated by $\mathfrak{n}_+$. Denote

$$\delta(\boldsymbol{\Lambda}, \boldsymbol{l}) \;=\; \dim\; \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{l}]\;.$$

It is well-known that for given $\boldsymbol{l}$ and a generic set of weights $\boldsymbol{\Lambda}$, we have

$$d(n-1, \boldsymbol{l}) \;=\; \delta(\boldsymbol{\Lambda}, \boldsymbol{l})\;.$$

For given $\boldsymbol{\Lambda}$ and $\boldsymbol{l}$, introduce the weight

$$\Lambda_\infty \;=\; \sum_{s=1}^{n} \Lambda_s \;-\; \sum_{i=1}^{r} l_i \alpha_i \;\in\; \mathfrak{h}^*$$

and the sequences of numbers $\boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$, where

$$\boldsymbol{m}_s \;=\; \{\, m_{s,1}, \dots, m_{s,r+1}\,\} \qquad \text{and} \qquad m_{s,i} \;=\; \langle\, \Lambda_s\,,\, e_{i,i}\,\rangle\;.$$

Having sequences $\boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$, we can recover $\boldsymbol{\Lambda}$ and $\boldsymbol{l}$ as follows:

$$\Lambda_s \;=\; \sum_{i=1}^{r+1} m_{s,i}\, e^*_{i,i} \qquad \text{and} \qquad \sum_{i=1}^{r} l_i\, \alpha_i \;=\; \sum_{s=1}^{n} \Lambda_s \;-\; \Lambda_\infty\;. \tag{3}$$

We say that the set of sequences $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ of complex numbers is *admissible*, if all of the numbers $\boldsymbol{l} = (l_1, \ldots, l_r)$, defined by (3), are nonnegative integers.

2.2. **Master functions.** Let $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a point with distinct coordinates. Let

$$\boldsymbol{\Lambda} \;=\; (\Lambda_1, \ldots, \Lambda_n) \;, \qquad \Lambda_s \in \mathfrak{h}^* \;,$$

be a collection of $\mathfrak{gl}_{r+1}$-weights and $\boldsymbol{l} = (l_1, \ldots, l_r)$ a collection of nonnegative integers. Set $l = l_1 + \cdots + l_r$. Introduce a function of l variables

$$\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{l_1}^{(1)}, \ldots, t_1^{(r)}, \ldots, t_{l_r}^{(r)})$$

by the formula

$$\Phi(\boldsymbol{t}; \boldsymbol{z}; \boldsymbol{\Lambda}; \boldsymbol{l}) = \tag{4}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{l_i} \prod_{s=1}^{n} (t_j^{(i)} - z_s)^{-(\Lambda_s, \alpha_i)} \prod_{i=1}^{r} \prod_{1 \le j < s \le l_i} (t_j^{(i)} - t_s^{(i)})^2 \prod_{i=1}^{r-1} \prod_{j=1}^{l_i} \prod_{k=1}^{l_{i+1}} (t_j^{(i)} - t_k^{(j+1)})^{-1} \;.$$

The function Φ is a (multi-valued) function of $\boldsymbol{t}$, depending on parameters $\boldsymbol{z}, \boldsymbol{\Lambda}$. The function is called *the master function*.

The master functions arise in the hypergeometric solutions of the KZ equations [**?**, V1] and in the Bethe ansatz method for the Gaudin model [RV, ScV, MV1, MV2, MV3, V2, MTV].

The product of symmetric groups

$$S_{\boldsymbol{l}} \;=\; S_{l_1} \times \cdots \times S_{l_r}$$

acts on variables $\boldsymbol{t}$ by permuting the coordinates with the same upper index. The master function is $S_{\boldsymbol{l}}$-invariant.

A point $\boldsymbol{t}$ with complex coordinates is called *a critical point* of $\Phi(\,\cdot\,; \boldsymbol{z}; \boldsymbol{\Lambda}; \boldsymbol{l})$ if the following system of l equations is satisfied

$$\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_1)}{t_j^{(1)} - z_s} - \sum_{s=1,\ s \ne j}^{l_1} \frac{2}{t_j^{(1)} - t_s^{(1)}} + \sum_{s=1}^{l_2} \frac{1}{t_j^{(1)} - t_s^{(2)}} = 0 \;, \tag{5}$$

$$\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_i)}{t_j^{(i)} - z_s} - \sum_{s=1,\ s \ne j}^{l_i} \frac{2}{t_j^{(i)} - t_s^{(i)}} + \sum_{s=1}^{l_{i-1}} \frac{1}{t_j^{(i)} - t_s^{(i-1)}} + \sum_{s=1}^{l_{i+1}} \frac{1}{t_j^{(i)} - t_s^{(i+1)}} = 0 \;,$$

$$\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_r)}{t_j^{(r)} - z_s} - \sum_{s=1,\ s \ne j}^{l_r} \frac{2}{t_j^{(r)} - t_s^{(r)}} + \sum_{s=1}^{l_{r-1}} \frac{1}{t_j^{(r)} - t_s^{(r-1)}} = 0 \;,$$

where $j = 1, \ldots, l_1$ in the first group of equations, $i = 2, \ldots, r-1$ and $j = 1, \ldots, l_i$ in the second group of equations, $j = 1, \ldots, l_r$ in the last group of equations.

In other words, a point $\boldsymbol{t}$ is a critical point if

$$\left(\Phi^{-1}\frac{\partial\Phi}{\partial t_j^{(i)}}\right)(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l}) \;=\; 0\ , \qquad i=1,\dots,r,\ j=1,\dots l_i\ .$$

In the Gaudin model, equations (5) are called *the Bethe ansatz equations.*

The set of critical points is $S_{\boldsymbol{l}}$-invariant.

2.3. **The case of isolated critical points.** We say that the pair $\boldsymbol{\Lambda},\boldsymbol{l}$ is *separating* if

$$(2\Lambda_\infty \;+\; \sum_{i=1}^r c_i\alpha_i,\ \sum_{i=1}^r c_i\alpha_i) \;+\; 2\sum_{i=1}^r c_i \;\neq\; 0$$

for all sets of integers $\{c_1,\dots,c_r\}$ such that $0\le c_i\le l_i,\ \sum_i c_i\neq 0$.

For example, if Λ_∞ is dominant integral, then $\boldsymbol{\Lambda},\boldsymbol{l}$ is separating.

Lemma 2.1 (Theorem 16 [ScV], Lemma 2.1 [MV2])**.** *If the pair $\boldsymbol{\Lambda},\boldsymbol{l}$ is separating, then the set of critical points of the master function $\Phi(\,\cdot\,;\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l})$ is finite.*

By Lemma 2.1, for given $\boldsymbol{l}$ and generic $\boldsymbol{\Lambda}$, the master function $\Phi(\,\cdot\,;\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l})$ has finitely many critical points.

Theorem 2.2. *Assume that the pair $\boldsymbol{\Lambda},\boldsymbol{l}$ is separating, then the number of $S_{\boldsymbol{l}}$-orbits of critical points counted with multiplicities is not greater than $d(n-1,\boldsymbol{l})$.*

The multiplicity of a critical point $\boldsymbol{t}$ is the multiplicity of $\boldsymbol{t}$ as a solution of system (5).

Proof. It is shown in [BMV] that, if $\Lambda_1,\dots,\Lambda_n,\Lambda_\infty$ is a collection of dominant integral weights, then the number of $S_{\boldsymbol{l}}$-orbits of critical points counted with multiplicities is not greater than $\delta(\boldsymbol{\Lambda},\boldsymbol{l})$. Together with equality $d(n-1,\boldsymbol{l})=d(\boldsymbol{\Lambda},\boldsymbol{l})$ for generic $\boldsymbol{\Lambda}$, this proves the theorem. ☐

3. Differential operators with quasi-polynomial flags of solutions

3.1. **Fundamental differential operator.** For the $S_{\boldsymbol{l}}$-orbit of a critical point $\boldsymbol{t}$ of the master function $\Phi(\,\cdot\,;\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l})$, define the tuple $\boldsymbol{y}^{\boldsymbol{t}}=(y_1,\dots,y_r)$ of polynomials in variable x,

$$y_i(x) \;=\; \prod_{j=1}^{l_i}(x-t_j^{(i)})\ , \qquad i=1,\dots,r\ .$$

Since $\boldsymbol{t}$ is a critical point, all fractions in (5) are well-defined. Therefore, the tuple $\boldsymbol{y}^{\boldsymbol{t}}$ has the following properties:

(6) Every polynomial y_i has no multiple roots.

(7) Every pair of polynomials y_i and y_{i+1} has no common roots.

(8) For every $i=1,\dots,r$ and $s=1,\dots,n$, if $m_{s,i}-m_{s,i+1}\neq 0$, then $y_i(z_s)\neq 0$.

A tuple of monic polynomials $(y_1,\dots,y_r)$ with properties (6)-(8) will be called *off-diagonal*.

Define quasi-polynomials $T_1,\dots,T_{r+1}$ in x by the formula

$$T_i(x) \;=\; \prod_{s=1}^{n} (x-z_s)^{-m_{s,i}} \;. \tag{9}$$

Consider the linear differential operator of order $r+1$,

$$D_{\boldsymbol{t}} = (\frac{d}{dx} - \ln'(\frac{T_{r+1}}{y_r}))\ (\frac{d}{dx} - \ln'(\frac{y_rT_r}{y_{r-1}}))\dots(\frac{d}{dx} - \ln'(\frac{y_2T_2}{y_1}))\ (\frac{d}{dx} - \ln'(y_1T_1))$$

where $\ln'(f)$ denotes $(df/dx)/f$ for any f. We say that $D_{\boldsymbol{t}}$ is *the fundamental operator* of the critical point $\boldsymbol{t}$.

Theorem 3.1. *Assume that the pair $\boldsymbol{\Lambda}$, $\boldsymbol{l}$ is separating. Then*

(i) *All singular points of $D_{\boldsymbol{t}}$ are regular and lie in $z_1,\dots,z_n,\infty$. The exponents of $D_{\boldsymbol{t}}$ at z_s are*

$$-m_{s,1}\,,\ -m_{s,2}+1\,,\ \dots\ ,\ -m_{s,r+1}+r$$

for $s=1,\dots,n$, and the exponents of $D_{\boldsymbol{t}}$ at ∞ are

$$m_{\infty,1}\,,\ m_{\infty,2}-1\,,\ \dots\ ,\ m_{\infty,r+1}-r\ .$$

(ii) *The differential equation $D_{\boldsymbol{t}}\,u \;=\; 0$ has solutions $u_1,\dots,u_{r+1}$ such that*

$$u_1 \;=\; y_1T_1$$

and for $i=2,\dots,r+1$ we have

$$\mathrm{Wr}\,(u_1,\dots,u_i) \;=\; y_i\ T_i\ T_{i-1}\ \dots\ T_1$$

where $\mathrm{Wr}\,(u_1,\dots,u_i)$ *denotes the Wronskian of $u_1,\dots,u_i$ and $y_{r+1}=1$.*

Proof. Part (ii) follows from the presentation of $D_{\boldsymbol{t}}$ as a product. To prove part (i) consider the operator $\tilde D_{\boldsymbol{t}} = T_1^{-1}\cdot D_{\boldsymbol{t}}\cdot T_1$, the conjugate of $D_{\boldsymbol{t}}$ by the operator of multiplication by T_1. Then

$$\tilde D_{\boldsymbol{t}} = (\frac{d}{dx} - \ln'(\frac{T_{r+1}}{y_rT_1}))\ (\frac{d}{dx} - \ln'(\frac{y_rT_r}{y_{r-1}T_1}))\dots(\frac{d}{dx} - \ln'(\frac{y_2T_2}{y_1T_1}))\ (\frac{d}{dx} - \ln'(y_1))\ .$$

It is enough to show that

(iii) All singular points of $\tilde D_{\boldsymbol{t}}$ are regular and lie in $\{z_1,\dots,z_n,\infty\}$.

(iv) The exponents of $\tilde D_{\boldsymbol{t}}$ at z_s are

$$0\,,\ m_{s,1}-m_{s,2}+1\,,\ \dots\ ,\ m_{s,1}-m_{s,r+1}+r\ ,$$

for $s=1,\dots,n$, and the exponents of $\tilde D_{\boldsymbol{t}}$ at ∞ are

$$-l_1\,,\ -m_{\infty,1}+m_{\infty,2}-1-l_1\,,\ \dots\ ,\ -m_{\infty,1}+m_{\infty,r+1}-r-l_1\ .$$

If all highest weights in the collection $\Lambda_1, \ldots, \Lambda_n, \Lambda_\infty$ are integral dominant, then statements (iii-iv) are proved in [MV2], Section 5. Hence statements (iii-iv) hold for arbitrary separating $\Lambda_1, \ldots, \Lambda_n$, $\boldsymbol{l}$. $\square$

3.2. **Quasi-polynomial flags.** Let $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a point with distinct coordinates. Let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ be an admissible set of sequences of complex numbers as in Section 2.1. Let

$$D \;=\; \frac{d^{r+1}}{dx^{r+1}} \;+\; A_1(x)\frac{d^r}{dx^r} \;+\; \ldots \;+\; A_r(x)\frac{d}{dx} \;+\; A_{r+1}(x)$$

be a differential operator with rational coefficients.

We say that D *is associated with* $\boldsymbol{z}$, $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$, if

- All singular points of D are regular and lie in $z_1, \ldots, z_n, \infty$.
- The exponents of D at z_s are $-m_{s,1}$, $-m_{s,2}+1$, $\ldots$, $-m_{s,r+1}+r$ for $s = 1, \ldots, n$.
- The exponents of D at ∞ are $m_{\infty,1}$, $m_{\infty,2} - 1$, $\ldots$, $m_{\infty,r+1} - r$.

Define quasi-polynomials $T_1, \ldots, T_{r+1}$ in x by formula (9).

We say that an operator D, associated with $\boldsymbol{z}$, $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$, *has a quasi-polynomial flag*, if there exists a tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$ of monic polynomials in x, such that

(i) For $i = 1, \ldots, r$, $\deg\, y_i \,=\, l_i$.
(ii) The tuple $\boldsymbol{y}$ is off-diagonal in the sense of (6)-(8).
(iii) The differential equation $D\,u \,=\, 0$ has solutions $u_1, \ldots, u_{r+1}$ such that

$$u_1 \;=\; y_1 T_1$$

and for $i = 2, \ldots, r+1$ we have

$$\mathrm{Wr}\,(u_1, \ldots, u_i) \;=\; y_i\; T_i\; T_{i-1}\; \ldots\; T_1\;,$$

where $y_{r+1} = 1$.

Proposition 3.2. *An operator D, associated with $\boldsymbol{z}$, $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$, has a quasi-polynomial flag if and only there exists a tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$ of monic polynomials in x with properties (i-ii) and*

(iv) *The differential operator D can be presented in the form*

$$D = (\frac{d}{dx} - \ln'(\frac{T_{r+1}}{y_r}))\;(\frac{d}{dx} - \ln'(\frac{y_r T_r}{y_{r-1}}))\ldots(\frac{d}{dx} - \ln'(\frac{y_2 T_2}{y_1}))\;(\frac{d}{dx} - \ln'(y_1 T_1))\;.$$

Proof. The equivalence of (iii) and (iv) follows from the next lemma.

Lemma 3.3. *Let $u_1, \ldots, u_{r+1}$ be any functions such that*

$$\mathrm{Wr}\,(u_1, \ldots, u_i) \;=\; y_i\; T_i\; T_{i-1}\; \ldots\; T_1\;,$$

for $i = 1, \dots, r+1$. Then $u_1, \dots, u_{r+1}$ form a basis of the space of solutions of the equation $Du = 0$ with

$$D = (\frac{d}{dx} - \ln'(\frac{T_{r+1}}{y_r}))\ (\frac{d}{dx} - \ln'(\frac{y_r T_r}{y_{r-1}})) \dots (\frac{d}{dx} - \ln'(\frac{y_2 T_2}{y_1}))\ (\frac{d}{dx} - \ln'(y_1 T_1))\ .$$

Proof. It is enough to prove that $D\, u_i \ = \ 0$ for $i = 1, \dots, r+1$. By induction on i, we obtain that

$$(\frac{d}{dx} - \ln'(\frac{y_i T_i}{y_{i-1}})) \dots (\frac{d}{dx} - \ln'(\frac{y_2 T_2}{y_1}))\ (\frac{d}{dx} - \ln'(y_1 T_1))\ u \ = \ \frac{W(u_1, \dots, u_i, u)}{y_i\, T_i\, T_{i-1}\, \dots\, T_1}$$

from which the statement follows. □

□

Examples of differential operators with quasi-polynomial flags are given by Theorem 3.1. If $\boldsymbol{t}$ is a critical point of the master function $\Phi(\,\cdot\,; \boldsymbol{z}; \boldsymbol{\Lambda}; \boldsymbol{l})$, then by Theorem 3.1, the fundamental differential operator $D_{\boldsymbol{t}}$ is associated with $\boldsymbol{z},\ \boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ and has a quasi-polynomial flag.

Having $\boldsymbol{z},\ \boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$, define $\boldsymbol{\Lambda}, \boldsymbol{l}$ by (3) and the master function $\Phi(\,\cdot\,; \boldsymbol{z}; \boldsymbol{\Lambda}; \boldsymbol{l})$ by (4).

Having a tuple $\boldsymbol{y} = (y_1, \dots, y_r),\ \ y_i(x) \ = \ \prod_{j=1}^{l_i} (x - t_j^{(i)})$, denote by

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(r)}, \dots, t_{l_r}^{(r)}) \tag{10}$$

a point in $\mathbb{C}^l$, whose coordinates are roots of the polynomials of $\boldsymbol{y}$. The tuple $\boldsymbol{y}$ uniquely determines the $S_{\boldsymbol{l}}$-orbit of $\boldsymbol{t}$.

Theorem 3.4. *Assume that an operator D is associated with $\boldsymbol{z},\ \boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ and has a quasi-polynomial flag. Let $\boldsymbol{y}$ be the corresponding tuple of polynomials. Then the point $\boldsymbol{t}$, introduced in (10), is a critical point of the master function $\Phi(\,\cdot\,; \boldsymbol{z}; \boldsymbol{\Lambda}; \boldsymbol{l})$.*

Proof. Let $u_1, \dots, u_{r+1}$ be the set of solutions of the differential equation $D\, u \ = \ 0$ giving the quasi-polynomial flag. Introduce the functions $\tilde y_1, \dots, \tilde y_r$ by the formulas $u_2 = \tilde y_1 T_1$ and for $i = 2, \dots, r$,

$$\tilde y_i\, T_i\, T_{i-1}\, \dots\, T_1 \ = \ \mathrm{Wr}\,(u_1, \dots, u_{i-1}, u_{i+1})\ .$$

The functions $\tilde y_1, \dots, \tilde y_r$ are multi-valued functions.

- Singularities of all branches of all of these functions lie in $z_1, \dots, z_n$.

It follows from the Wronskian identities of [MV2], that

- $\mathrm{Wr}\,(\tilde y_1, y_1) \ = \ \dfrac{T_2}{T_1}\, y_2\ , \quad \mathrm{Wr}\,(\tilde y_r, y_r) \ = \ \dfrac{T_{r+1}}{T_r}\, y_{r-1}\ , \quad \mathrm{Wr}\,(\tilde y_i, y_i) \ = \ \dfrac{T_{i+1}}{T_i}\, y_{i-1}\, y_{i+1}$

for $i = 2, \dots, r-1$. These two properties of functions $\tilde{y}_1, \dots, \tilde{y}_r$ imply equations (5) for roots of polynomials $y_1, \dots, y_r$, see [MV2]. This shows that the point $\boldsymbol{t}$ is a critical point. $\square$

The correspondence between critical points and differential operators with quasi-polynomial flags is reflexive. If $\boldsymbol{t}$ is a critical point of the master function $\Phi(\,\cdot\,; \boldsymbol{z}; \boldsymbol{\Lambda}; \boldsymbol{l})$, which corresponds by Theorem 3.4 to the differential operator D, then D is the fundamental differential operator of the critical point $\boldsymbol{t}$.

3.3. **Conclusion.** Let $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ be a point with distinct coordinates. Let $\boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ be an admissible set of sequences of complex numbers as in Section 2.1. Define $\Lambda_1, \dots, \Lambda_n, \Lambda_\infty$, $\boldsymbol{l}$ by (3). Assume that $\boldsymbol{l}$ is fixed and $\Lambda_1, \dots, \Lambda_n$ are separating (that is generic). Then the number of differential operators D, associated with $\boldsymbol{z}$, $\boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ and having a quasi-polynomial flag, is not greater than $d(n-1, \boldsymbol{l})$, see Theorems 2.2 and 3.4.

It is interesting to note that if $\Lambda_1, \dots, \Lambda_n, \Lambda_\infty$, are dominant integral, then the number of differential operators D, associated with $\boldsymbol{z}$, $\boldsymbol{m}_1, \dots, \boldsymbol{m}_{r+1}, \boldsymbol{m}_\infty$ and having a quasi-polynomial flag, is not greater than the multiplicity $\delta(\boldsymbol{\Lambda}, \boldsymbol{l})$, which could be a smaller number. See Theorem 3.4 and description in [MV2] of the critical points of the corresponding master functions.

References

[BMV] P. Belkale, E. Mukhin, and A. Varchenko, *Multiplicity of critical points of master functions and Schubert calculus* Preprint (2005), 1–23; `math.QA/0503132`

[BBS] J. Borcea, P. Branden, and B. Shapiro, *On higher Lame equations, Heine-Stieltjes and Van Vleck polynomials*, in preparation

[H] E. Heine, *Handbuch der Kugelfunctionen*, G. Reimer Verlag, Berlin, vol. 1 (1878), 472–479

[MTV] E. Mukhin, V. Tarasov, and A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Preprint (2005), 1–17, `math.AG/0512299`

[MV1] E. Mukhin and A. Varchenko, *Remarks on critical points of phase functions and norms of Bethe vectors*, Adv. Stud. Pure Math. **27** (2000), 239–246

[MV2] E. Mukhin and A. Varchenko, *Critical Points of Master Functions and Flag Varieties*, Communications in Contemporary Mathematics **6** (2004), no. 1, 111-163

[MV3] E. Mukhin and A. Varchenko, *Norm of a Bethe Vector and the Hessian of the Master Function*, Compos. Math. **141** (2005), no. 4, 1012–1028

[RV] N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, Geometry, Topology and Physics for R. Bott, Intern. Press, 1995, 293–322

[SV] V V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), 139-194

[ScV] I. Scherbak and A. Varchenko, *Critical points of functions, $\mathfrak{sl}_2$ representations, and Fuchsian differential equations with only univalued solutions*, Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday, Mosc. Math. J. **3** (2003), no. 2, 621–645

[St] T. Stieltjes, *Sur certains polynomes qui verifient une equation differentielle lineaire du second ordre et sur la theorie des fonctions de Lame*, Acta Math. **8** (1885), 321–326

[Sz] G. Szego, *Orthogonal Polynomials*, AMS, 1939.

[V1] A. Varchenko, *Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*, Advanced Series in Mathematical Physics, Vol. 21, World Scientific, 1995

[V2] A. Varchenko, *Bethe Ansatz for Arrangements of Hyperplanes and the Gaudin Model*, Preprint (2004), 1–17; `hep-th/0408001`

[V3] A. Varchenko, *Critical points of the product of powers of linear functions and families of bases of singular vectors*, Compositio Mathematica **97** (1995), 385–401